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B.Sc. DEGREE (C.B.C.S.S.) EXAMINATION, APRIL 2022

Fifth Semester

Core Course—ABSTRACT ALGEBRA

(Common for Model I and Model II B.Sc. Mathematics)

[2013 to 2016 Admissions]

Time : Three Hours Maximum Marks : 80

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Define * on Q^+ by $a * b = \dfrac{ab}{2}$. Find the inverse of a.
2. Define cyclic group.
3. Find the number of elements in the symmetric group S_n of n elements.
4. Find the number of generators of cyclic group of order 12.
5. Define simple group.
6. Find the order of the factor group $Z_4 \times Z_{12}/\langle(2,2)\rangle$.
7. Describe all units in the ring Z_5.
8. Define integral domain.
9. Find the characteristic of the ring of integers Z.
10. Every ideal in a ring is a subring of the ring. State True or False.

$(10 \times 1 = 10)$

Part B

Answer any **eight** *questions.*
Each question carries 2 *marks.*

11. Let G be a group and H and K are subgroups of a group G. Prove that $H \cap K$ is a subgroup of G.
12. Let G be a group with binary operation $*$. Prove that $a * b = a * c$ implies $b = c$ and $b * a = c * a$ implies $b = c$ for $a, b, c \in G$.
13. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} \in S_8$. Express σ as a product of tranpositions.

Turn over

14. Find the number of elements in the cyclic subgroup of Z_{42} generated by 30.

15. For each $g \in G$. Define $i_g : G \to G$ by $xi_g = g^{-1}xg$. Prove that i_g is an automorphism of G.

16. Prove that every subgroup of an abelian group is normal.

17. Let ϕ be a homomorphism of a group G into a group G′, Let H be a normal subgroup of G. Prove that $H\phi$ is a normal subgroup of G′.

18. Find all solutions of the equation $x^3 - 2x^2 - 3x = 0$ in Z_{12}.

19. If p is a prime prove that Z_p is a field.

20. Let R be a ring with unity 1. Prove that R has characteristic $n > 0$ if and only if n is the smallest positive integer such that $n \, . \, 1 = 0$.

21. Prove that a field contains no proper non-trivial ideals.

22. Prove that nZ is an ideal of Z.

(8 × 2 = 16)

Part C

Answer any **six** *questions.*
Each question carries 4 marks.

23. Prove that in a group G, the identity and inverses are unique.

24. Let S be the set of all real numbers except – 1. Define * on S by $a * b = a + b + ab$.
    (a) Show that $*$ is a binary operation.
    (b) Show that (S, $*$) is a group.
    (c) Find the solution of the equation $2 * x * 3 = 7$ in S.

25. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$, $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$ be two permutations in S_6. Compute $\sigma\tau$ and $\sigma\tau^2$.

26. Let G and G′ be two groups and ϕ be an isomorphism from G to G′. Prove that ϕ maps identity onto identity and inverses onto inverses.

27. Prove that Z and Q both under addition are not isomorphic.

28. Let H be a subgroup of a group G. Prove that the relation $a \equiv_r b \pmod{H}$ if and only if $a^{-1}b \in H$ is an equivalence relation.

29. Let H be a subgroup of a group G. Prove that the operation of induced multiplication is well defined on the left (right) cosets of H if and only if every left coset is a right coset.

30. Prove that $\phi : Z \to R$ under addition defined by $n\phi = n$. Also find image and kernel of ϕ.

31. Prove that every finite integral domain is a field.

(6 × 4 = 24)

Part D

*Answer any **two** questions.*
Each question carries 15 marks.

32. (a) List the elements in the dihedral group D_4.

    (b) Construct the multiplication table for elements in D_4.

    (c) Find all subgroups of D_4 of order 2.

33. Find all subgroups of Z_{12} and give their lattice diagram.

34. State and prove Cayley's theorem.

35. (a) State and prove fundamental theorem of homomorphism.

    (b) Prove that $R = \{a + b\sqrt{2} / a, b \in Q\}$ with usual addition and multiplication is a ring.

$(2 \times 15 = 30)$